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Abstract. The recent availability of complete three dimensional samples of galaxies and clusters permits a direct study of their spatial properties. We present a brief review of galaxy correlations based on the methods of modern statistical Physics. These methods which are able to identify self-similar and non-analytical properties, allow us to test the usual homogeneity assumption of luminous matter distribution.

We conclude that both the three dimensional properties, and the angular $\log N - \log S$ relation, point out to the fact that the distribution of galaxies and clusters is fractal with $D \approx 2$ up to the deepest scale probed for luminous matter ($\gtrsim 1000h^{-1}Mpc$). This result has important implications for the theoretical framework that should be adopted.

1 Introduction

The first galaxy catalogues were only angular, namely they defined the two angles corresponding to the galaxy positions in the sky (Shane and Wirtanen, 1967). These angular distributions show structures at small scales but appear rather smooth at large angular scales. This situation was therefore fully satisfactory with respect to the theoretical expectations of a homogeneous universe (Peebles, 1980). In the late seventies however the first redshift measurements begun to appear and permitted the identification of the absolute distance of galaxies (Huchra et al., 1983). In this way it became possible to obtain the complete three dimensional distribution of galaxies. These distributions showed a more irregular structure with respect to the angular data with the appearance of superclusters and large voids. At first these large structures were considered as accidental or due to experimental incompleteness. But more and more data showed that the structures are all over and the voids do not fill with better measurements. This three dimensional picture did not show any more a clear tendency towards homogenization and was in contrast with the angular data. This controversial situation found an apparent solution with the first statistical analysis of the CfA1 galaxy catalogue (Davis & Peebles, 1983). In fact this analysis identified a small correlation length of only 5 Mpc in a catalogue that showed structures of much larger sizes. The statistical analysis appeared therefore to provide a way out such that large structures can be compatible with small correlation lengths.

In the following years the situation evolved in a dramatic way because deeper and deeper surveys showed larger and larger structures that appeared difficult to reconcile with such a small correlation length. In addition the catalogues of clusters gave a correlation length of 25 Mpc, five times larger than the one of galaxies, even though the clusters are made themselves of galaxies. This situation led to a great confusion and different authors looked for different possible solutions to the problem. At this stage various hypothesis were formulated like the luminosity segregation, the clustering richness relation that leads to the biased theory of structure formation etc. In the end the most popular picture is that clusters and galaxy structures require different theories because their correlation show different amplitudes. A linear theory is necessary for clusters while a non linear theory should be adopted for galaxies. The large structures can be compatible with small correlation lengths and with a large scale homogeneity because the amplitudes of the structures becomes smaller the larger is the structure. Finally a clear evidence of homogeneity cannot yet be obtained because the present samples are not yet fair.

In the past years, we have challenged this picture (Coleman & Pietronero, 1992) (Baryshev et al, 1994) by showing that it arises just by a mathematical inconsistency in the characterization of the galaxy and cluster correlations. Our main result is that a correct analysis of

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the data shows fractal correlations up to the present observational limits. The galaxy-cluster mismatch disappears and the visible universe is characterized by a multifractal distribution of matter when the galaxy masses are also included. This requires a radical change of perspective for the properties of the universe and for the theoretical methods that one should use to describe it.

In this lecture we present a colloquial discussion of these subjects including the most recent results (Sylos Labini et al., 1996a) (Sylos Labini et al., 1996b) (Sylos Labini et al., 1997).

2 The Space correlations analysis

The distribution of galaxies in space has been investigated very intensively in these last years. Several recent galaxy redshift surveys such as CfA 1 (Huchra et al., 1983), CfA 2 (De Lapparent et al., 1988) (Da Costa et al., 1994) (Park et al., 1994), SSRS1 (Da Costa et al., 1988), SSRS2 (Da Costa et al., 1994), Perseus Pisces (Haynes and Giovanelli, 1988), Las Campanas Redshift Survey (Schectman et al., 1996), APM-Stromlo (Loveday et al., 1992), IRAS catalogs (Strauss et al., 1992) (Fisher et al., 1995), pencil beams surveys (Broadhurst et al., 1990) (Bellanger and De Lapparent, 1995), and ESP (Vettolani et al., 1994) have uncovered remarkable structures such as filaments, sheets, superclusters and voids. These galaxy catalogues probe scales from $\sim 100 - 200h^{-1}Mpc$ for the wide angle surveys, up to $\sim 1000h^{-1}Mpc$ for the deeper pencil beam surveys (that cover a very narrow solid angle) and show that the Large Scale Structures (LSS) are the characteristic features of the visible matter distribution. One of the most important issues raised by these catalogues is that the scale of *the largest inhomogeneities* is comparable with *the extent of the surveys* in which they are detected. Hence from these data a new picture emerges in which the scale of homogeneity seems to shift to a very large value, not still identified.

The usual correlation analysis is performed by the $\xi(r)$ function (Peebles, 1980) defined as

$$\xi(r) = \frac{\langle n(r+r_0)n(r_0)\rangle}{\langle n\rangle^2} - 1 \qquad (1)$$

where $\langle n\rangle$ is the average density in the sample considered. Such an analysis leads to the identification of the "correlation length" $r_0 \approx 5h^{-1}Mpc$ (Davis & Peebles, 1983), defined as the distance where $\xi(r_0) \equiv 1$. This result appears incompatible with the existence of LSS of order of $50 - 200h^{-1}Mpc$. In fact, according to this result, the distribution should become smooth and regular at distances larger than r_0 (say for example $\sim 2 \div 3r_0 \sim 15h^{-1}Mpc$), while it is evident that this is not the case. The main problem of the $\xi(r)$-analysis is that it is based on the *assumption* that the distribution of galaxies has reached the real homogeneity within the considered samples. (We refer the reader to Coleman & Pietronero (1992) for a detailed discussion of this subject). The identification of the correlation length r_0 as the amplitude of $\xi(r)$ is actually not correct if the system has long range (power law) correlations. In fact, only in the case of a really homogenous sample the amplitudes of correlation acquire a physical meaning. In the opposite case and anyhow for the range of scales in which the structure is self-similar (even if homogeneity is eventually achieved at large scale) it is necessary to change the theoretical language and perspective and adopt the one that is appropriate for self-similar and non-analytical structures (Baryshev et al, 1994) (Sylos Labini et al., 1997) (Mandelbrot, 1982) (Pietronero and Tosatti, 1986) (Erzan et al., 1995).

The basic idea of our approach is to perform a correlation analysis that does not require any a priori assumption (Coleman & Pietronero, 1992). In particular the normalization of the correlation function to the average density, as used in the definition of $\xi(r)$ (eq.1) is avoided because in the case of irregular and non-analytical distributions, the average density may be not a well defined quantity. In fact, in the case of fractal correlations the average conditional density has a power law decay from *any occupied point of the structure* (Coleman & Pietronero, 1992) (Mandelbrot, 1982)

$$\Gamma(r) \sim \langle n(r+r_0)n(r_0)\rangle \sim r^{D-3} \qquad (2)$$

where D is the fractal dimension. In the case of a homogenous distribution $D = 3$ and $\Gamma(r) = \langle n\rangle$, and the average density is clearly well defined.

Following Coleman & Pietronero (1992) the expression of the $\xi(r)$ in the case of fractal distributions is: $\xi(r) = ((3-\gamma)/3)(r/R_s)^{-\gamma} - 1$ where R_s is the depth of the spherical volume where one computes the average density and $\gamma = 3 - D$ is the correlation exponent. From the previous expression it follows that

i.) the so-called correlation length r_0 (defined as $\xi(r_0) = 1$) is a linear function of the sample size R_s and hence it is a spurious quantity without physical meaning but it is simply related to the sample finite size *ii.)* $\xi(r)$ is power law only for $((3-\gamma)/3)(r/R_s)^{-\gamma} >> 1$ hence for $r \ll r_0$: for larger distances there is a clear deviation from the power law behaviour due to the definition of $\xi(r)$. This deviation, however, is just due to the size of the observational sample and does not correspond to any real change of the correlation properties. It is clear that if one estimates the exponent of $\xi(r)$ at distances $r \lesssim r_0$, one systematically obtains a higher value of the correlation exponent due to the break of $\xi(r)$ in the log-log plot. The fact that the fractal dimension has been estimated to be $D = 1.2$ (for example (Davis & Peebles, 1983)) is only due to the fact that its has been estimated by the $\xi(r)$ in the region of length scales $r \sim r_0$: this result is completely spurious and does not depend on the real properties of the distribution (Sylos Labini et al., 1997).

Coleman & Pietronero (1992) by analyzing the CfA1 redshift survey showed that galaxy distribution in this sample is fractal up to $\sim 20h^{-1}Mpc$. Recently we have extended such a result finding by the conditional density analysis (Eq.(3)) that galaxy distribution in several volume limited samples (e.g., Coleman & Pietronero, 1992) galaxy redshift catalogs such as has fractal long range correlations up to their limiting depth that is $\sim 150h^{-1}Mpc$ for Perseus-Pisces, LEDA, CfA1, SSRS1, Stromlo-APM, IRAS redshift surveys, ESP and LCRS. (Fig. 1). Moreover by analyzing the behaviour of the radial density from the vertex rather than the conditional density (Eq.(3)), we are able to conclude that galaxy distribution has fractal properties with $D \approx 2$ up to $\sim 1000h^{-1}Mpc$ (Sylos Labini et al., 1997).

In addition we have studied several cluster catalogs (Abell, ACO, APM) showing that galaxies and clusters are different representations of a single self-similar structure: the correlations of clusters appear to be the continuation of galaxy correlations at larger scales, and clusters have the same fractal dimension $D \approx 2$ (Coleman & Pietronero, 1992) (Montuori et al., 1997) of galaxies (Fig. 2).

In no case we observe any trend towards homogeneity. This implies that, within the present observations, no characteristic correlation length can be defined, and that the usual one r_0 is actually spurious.

This new analysis we have discussed here, reconciles the statistical studies with the observed Large Scale Structures. The existence of such LSS, whose extent is limited only by *the size of the surveys* in which they are detected is now compatible with the fact that the statistical analysis shows that such structures are characterized by long range correlations.

3 The $\log N - \log S$ relation

We now discuss other experimental evidences in addition to the correlation analysis (Sylos Labini et al., 1996b). It is clear that the most complete information about galaxy distribution comes from the full three dimensional samples, and that the angular catalogs have a poorer quantitative information, even if usually they contain a much larger number of galaxies. However, one of the most important tasks in observational astrophysics, is the determination of the $\log N - \log S$ relation for various kinds of objects: galaxies in the various spectral band (from ultraviolet to infrared), radio-galaxies, quasars, X-ray sources and γ-ray bursts. This relation gives the number $N(S)$ (integral or differential) of objects, for unit solid angle, with *apparent flux* larger than a certain limit S. The determination of such a quantity avoids the measurements of the distance, which is always a very complex task.

Let us consider this problem in more detail. We assume as work hypothesis (that can be tested in the real data) that the number of objects per unit volume and unit *intrinsic* luminosity is given by

$$\nu(\vec{r}, L)d^3rdL = n(\vec{r})\phi(L)d^3rdL \qquad (3)$$

i.e. one can consider the total density ν in a space density $n(\vec{r})$ multiplied a luminosity function $\phi(L)$. As we are going to discuss briefly in the following, the galaxian luminosities are strongly correlated with their positions in space. This clear observational fact can be studied quantitatively with the multifractal formalism (Coleman & Pietronero, 1992) (Sylos Labini and Pietronero, 1996). However the approximation of Eq.3 can be considered valid for the purpose of the present discussion, and the multifractal properties will affect very weakly the following results (Sylos Labini and Pietronero, 1996) (Sylos Labini et al., 1996b). Neglecting the relativistic effects (that are usually model dependent (Baryshev et al, 1994, e.g.,)), the apparent luminosity S of an object of intrinsic luminosity L at distance R from the observer is given by

$$S = \frac{L}{4\pi R^2} \qquad (4)$$

For a fractal distribution the number of sources within a radius R is given by the so-called mass-length relation (Mandelbrot, 1982)

$$N(<R) = BR^D \qquad (5)$$

From eq.5 and eq.3, using eq.4, we have that the number of objects with apparent flux greater than S is given by

$$N(>S) \sim S^{-\frac{D}{2}} \qquad (6)$$

(for galaxies this relation is usually expressed in terms of magnitude (Sylos Labini et al., 1996b, e.g.,)) Hence eq.6 allows the determination of the fractal dimension *without* measuring the distance. Although this seems to be the most direct way to study the space properties, one should consider a very important limitation in the determination of the $\log N - \log S$ relation. It is possible to show that the genuine behavior described by eq.6 is reached only for faint apparent fluxes (Sylos Labini et al., 1996b). This problem is due to the fact that eq.6 is determined from a single point of observation and this relation is averaged over different directions in the sky but *not over different observers*: in this case the finite size effects may play an important role in the determination of an integrated quantity as $N(>S)$. The point is that a fractal structure is dominated by intrinsic fluctuations at all scales whose convolution leads to an average power law decay for the conditional density.

Usually the number counts relation is written in terms of the apparent magnitude m, $S \sim 10^{-0.4m}$ (note that bright galaxies correspond to small m). In terms of m, Eq.(7) becomes $\log N(<m) \sim \alpha m$ with $\alpha = D/5$ (Baryshev et al, 1994) (Peebles, 1993). In Fig.4 we have collected all the recent observations of $N(<m)$ versus

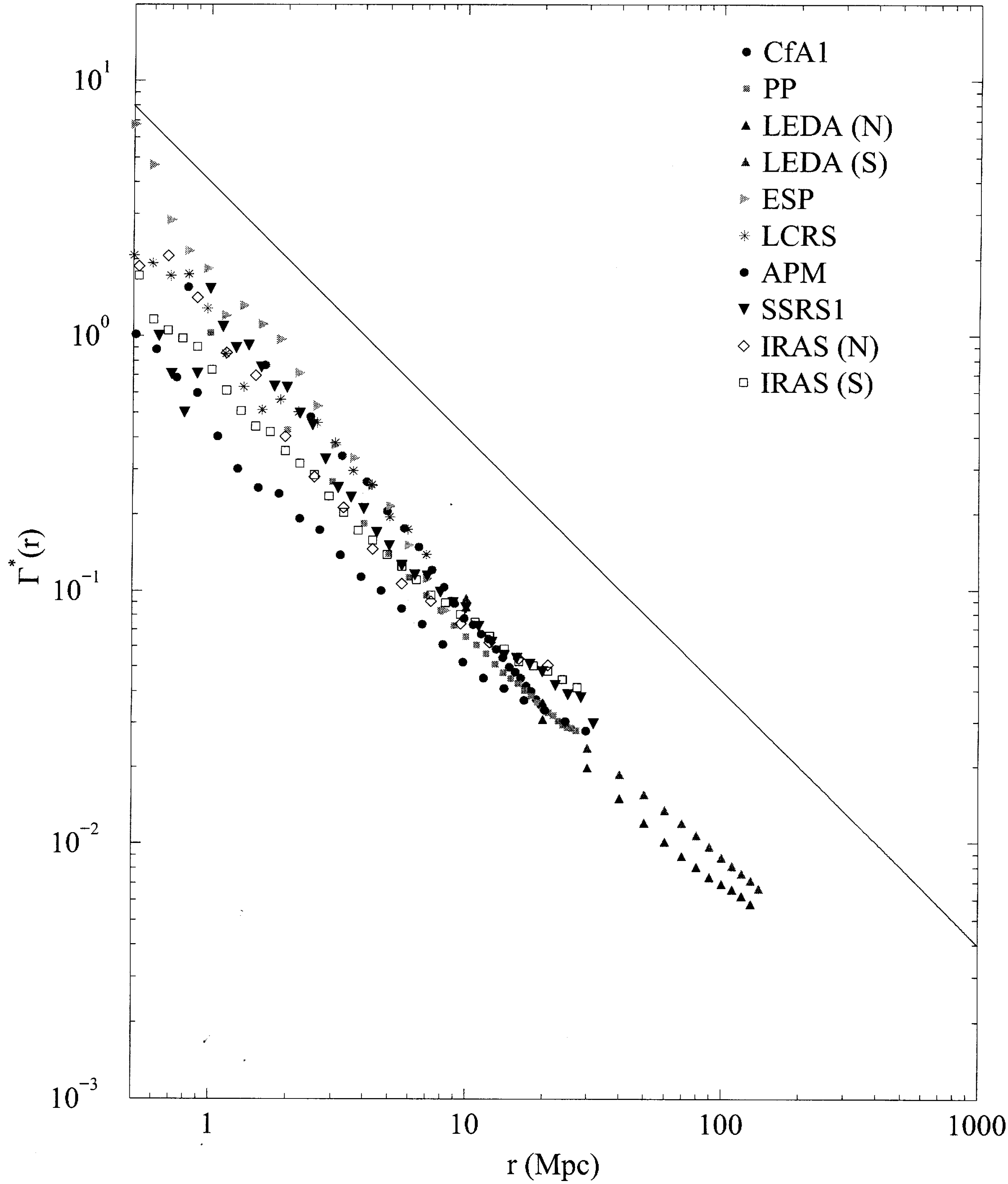

Fig. 1. The conditional average density eq.(2) for several volume limited samples of various redshift surveys: CfA1, Perseus-Pisces, LEDA APM-Stromlo, SSRS1, ESP, LCRS and the IRAS redshift surveys. The amplitude in the various cases is not arbitrary, and it is normalized only to take into account the different luminosity selection in the different samples (see Sylos Labini *et al.* 1997, for a detailed discussion of such a normalization). The reference line has a slope -1 that corresponds to a fractal dimension $D = 2$.

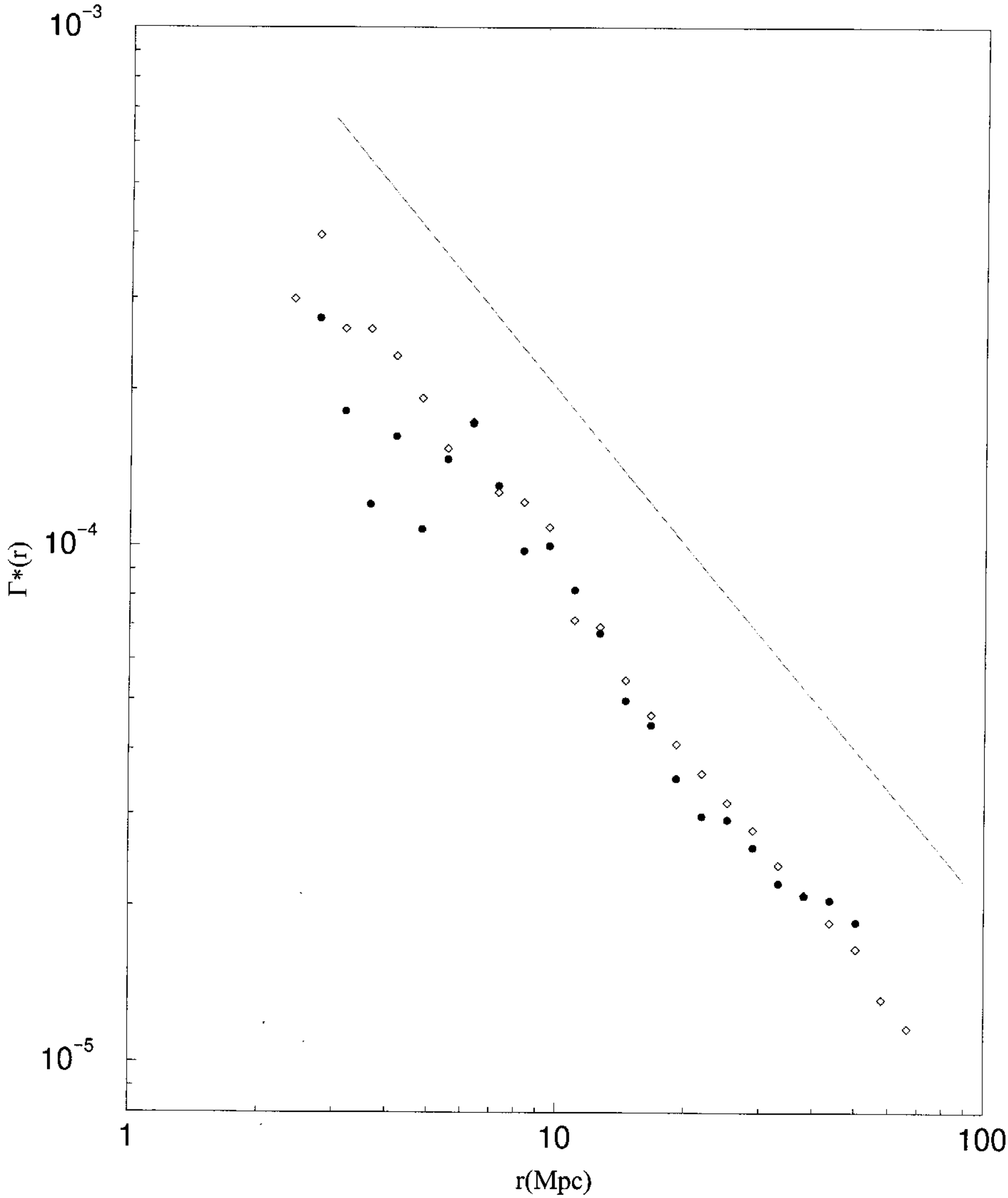

Fig. 2. The conditional average density (eq.2) for galaxy clusters: Abell and ACO. The reference line has a slope -1 that corresponds to a fractal dimension $D = 2$.

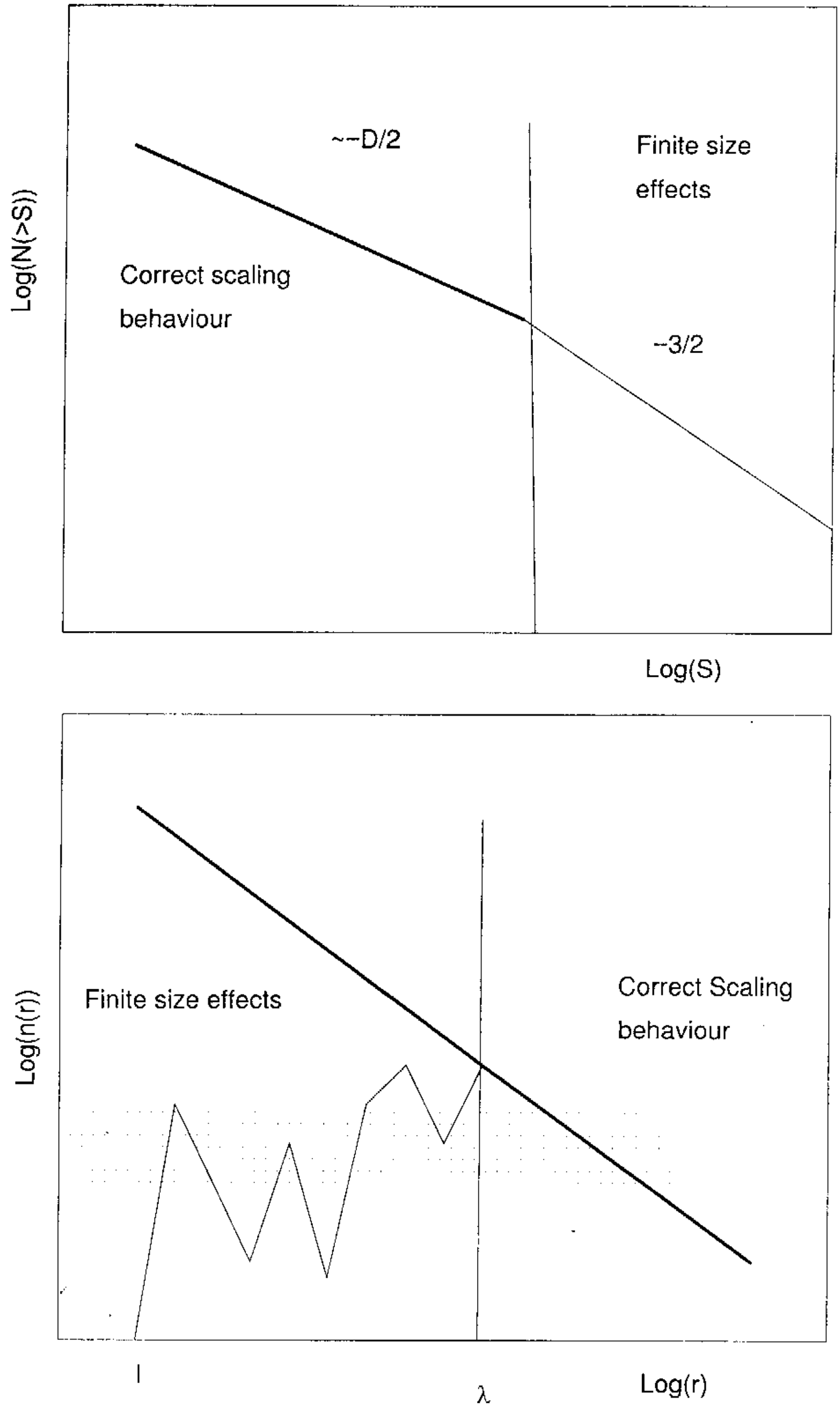

Fig. 3. *a* The behaviour of the conditional density computed from a single point in the ideal case of a fractal structure. Before the distance ℓ (that is of the order of the characteristic size of the Voronoi polyhedron) in average, one does not find any other points. Beyond this distance one sees a fluctuating region up to the scale λ that is related to the intrinsic properties of the fractal structure. Finally the correct scaling regime is reached. (*b*) The $N(>S)$ relation for the fractal structure whose density is shown in Fig.3. At faint fluxes, corresponding to large distances, one observes the correct scaling behaviour with an exponent $-D/2$, while at bright fluxes the finite size effects dominate the behaviour. In this case one detects an exponent $\gtrsim -3/2$ that seems to be in agreement with the homogenous case, but that is just due to the highly fluctuating behaviour of the density

m (Sylos Labini et al., 1996b). One can see that at small scales (small m) the exponent is $\alpha \approx 0.6$, while at larger scales (large m) it changes into $\alpha \approx 0.4$. The usual interpretation (Peebles, 1993) is that $\alpha \approx 0.6$ corresponds to $D \approx 3$ consistent with homogeneity, while at large scales galaxy evolution and space time expansion and galaxy evolution effects are invoked to explain the lower value $\alpha \approx 0.4$. On the basis of the previous discussion of the VL samples we can see that this interpretation is untenable. In fact, we know for sure that, at least up to $\sim 150h^{-1}Mpc$ there are fractal correlations so one would eventually expect the opposite behaviour. Namely small value of $\alpha \approx 0.4$ (consistent with $D \approx 2$) at small scales followed by a crossover to an eventual homogeneous distribution at large scales ($\alpha \approx 0.6$ and $D \approx 3$). The main problem in the study of the number counts relations, is that it is determined by only one observer. Namely, if one observes the conditional density from one point only, one observers a regime is characterized by a very fluctuating behavior that is just due to finite size effects and beyond this fluctuating region the correct scaling properties can be recovered (Fig.3). Correspondingly at bright apparent flux, that are associated to small distance of the sources, one expects to see an exponent $\gtrsim -3/2$, that seems to be in agreement with the homogeneous case. On the contrary this is just a spurious effect that arises from the fact that eq.6 is strongly affected by finite size fluctuations at small scales. At faint apparent fluxed, one is integrating the density in the correct scaling regime, and in this region the genuine statistical properties of the system can be detected (Fig.3b).

This behaviour is the general more and we have a double power law decay for the $\log N - \log S$ relation, one at Bright fluxes with exponent $\gtrsim -3/2$, and the other one at faint fluxes where the genuine behaviour is reached ($\sim -D/2$) (Sylos Labini et al., 1996b).

We have shown that the double power law behavior for the $\log N - \log S$ relation is verified for several objects: galaxies (in the blue, redshift, yellow, infrared and ultraviolet spectral bands - Fig.5), radio galaxies (at various wavelengths), quasars, X-ray sources and, finally, γ-ray bursts (Sylos Labini et al., 1997). Except the case of visible galaxies, where there are available complete redshift samples, for the other astrophysical objects the $\log N - \log S$ relation represents the only way of studying the spatial clustering. *In all these cases we find the same behavior*: an exponent $\gtrsim -3/2$ (that in terms of magnitudes means $\gtrsim D/5$) at bright fluxes, and an exponent $-D/2 \sim -1$ at faint fluxes, that corresponds to $D \approx 2$. Despite the experimental uncertainties, that in some cases can be important, we have that all these objects show the same tendency, i.e. they are all fully compatible with a fractal distribution in space (with the same fractal dimension) up to the deepest scales ever probed for luminous matter (Sylos Labini et al., 1997).

4 Luminosity and Space distributions

In the previous sections, we have discussed galaxy correlations in terms of a set of points corresponding to their position in space. Galaxies can be characterized by their luminosity (or mass). Therefore is possible to study the whole matter distribution, i.e. weighing each point by its mass. It is natural to consider the possible scale invariant properties of this distribution, and this requires a generalization of the fractal dimension and use the concept of multifractality (MF). The concept of MF is appropriate to discuss physical systems with local properties of self-similarity, in which the scaling properties are defined by a continuous distribution of exponents. This situation requires the generalization of the simple fractal scaling to a MF distribution in which a continuous set of exponents is necessary to describe the spatial scaling of peaks of different weight (mass or luminosity). In this respect the mass and space distributions become naturally entangled with each other.

A MF analysis of the CfA1 (Coleman & Pietronero, 1992)and Perseus-Pisces (Sylos Labini et al., 1996a) (Sylos Labini and Pietronero, 1996) (see also Garrido et al. (1996)) shows that also the full distribution is scale invariant and this lead to a new important relation between the luminosity function and the space correlations properties (Sylos Labini and Pietronero, 1996). In fact, the continuous set of exponents $[\alpha, f(\alpha)]$ that describes a MF distribution can characterize completely the galaxy distribution when one considers the mass (or luminosity) of galaxies in the analysis. In this way many observational evidences are linked together and arise naturally from the self-similar properties of the distribution. Considering a MF distribution, the usual power-law space correlation properties correspond just to a single exponent of the $f(\alpha)$ spectrum: such an exponent simply describes the space distribution of the support of the MF measure. Furthermore the shape of the luminosity function (LF), i.e. the probability of finding a galaxy of a certain luminosity per unit volume, is related to the $f(\alpha)$ spectrum of exponents of the MF (Sylos Labini and Pietronero, 1996).

These results have important consequences from a theoretical point of view. In fact, when one deals with self-similar structures the relevant physical phenomenon that leads to the scale-invariant structures is characterized by the *exponent* and *not the amplitude* of the physical quantities that characterizes such distributions.

5 Conclusions

The main assumption of modern cosmology, i.e. the homogeneity of matter distribution at some large scale is lacking a clear experimental support. On the contrary we have shown that galaxy distributions in all the available redshift surveys, present power law (fractal) long

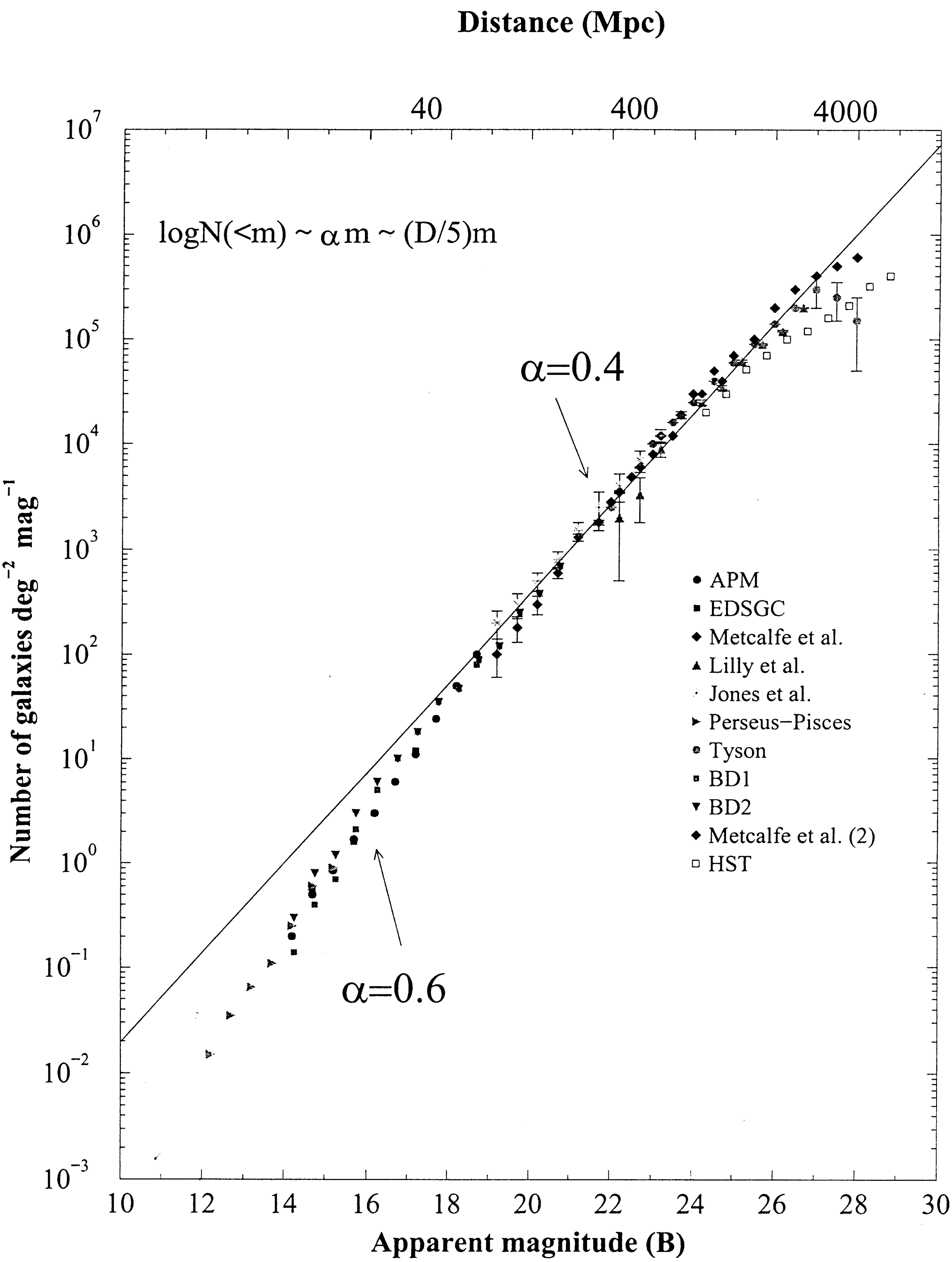

Fig. 4. The counts of galaxies as a function of the apparent magnitude m in the visible B-band. In the range $12 \div 19$, that corresponds to bright apparent fluxes, the counts (in terms of magnitude) show an exponent $\alpha = D/5 \approx 0.6$ that is usually interpreted as a sign of homogeneity ($D \approx 3$). In the range $m \sim 19 \div 28$ (corresponding to faint apparent fluxes) the exponent is lower $\alpha \approx 0.4$ and it is usually interpreted as an effect of galaxy evolutions or space-time expansion. Our studies show instead that the genuine exponent is $\alpha \approx 0.4$ in the entire range $12 \div 28$ consistent with a fractal structure is space with dimension $D \approx 2$, as obtained also from the complete spatial analysis. The value $\alpha \approx 0.6$ (that, in many other cases, is $\alpha \gtrsim 0.6$) at small scales is spurious and it is just due to finite size effects.

range correlations up to their limiting depth R_s, with $D \approx 2$. In particular we find that in various three dimensional surveys $R_s \approx 1000h^{-1}Mpc$, that corresponds to one forth of the Hubble radius (Sylos Labini et al., 1997).

Moreover we have discussed the reinterpretation of the so-called $\log N - \log S$ relation, that gives the number of objects with apparent flux greater than S. This is strongly related to the behaviour of the space density, and we find that, for various kind of astrophysical sources, it is fully compatible with a fractal distribution, with almost the same fractal dimension found by the whole three dimensional analysis (i.e. $D \approx 2$).

This picture unifies the various different observations of luminous matter in cosmology, without introducing any "ad hoc" hypotheses. This brings us to a new picture which simplifies or eliminate a number of inconsistencies of the standard one:

- the quite small "correlation length" ($r_0 \approx 5h^{-1}Mpc$) found by the $\xi(r)$ function analysis, that is in contrast with the existence of large scales structures extended over some hundreds $h^{-1}Mpc$, is spurious. This has allowed us to clarify also the mismatch galaxy-cluster, and points out that the clusters and the galaxies are different representations, of the same self-similar structure.

- Galaxy counts, (i.e the so-called $\log N - \log S$ relation). We have shown that the galaxy counts are completely compatible with the full three dimensional correlation analysis. Moreover we found that various other astrophysical objects such as radio-galaxies, quasars, X-ray sources and γ-ray bursts are completely compatible with a fractal structure in space with $D \approx 2$.

In summary, our result, the existence of a single fractal structure for the different kinds of astrophysical objects up to the deepest scales ever probed for luminous matter, clarifies various inconcistencies of observational astrophysics, and points out which are the real theoretical tasks that a consistent model should clarify.

In this new picture, the theoretical problem becomes clear even though probably more harder. The experimental facts are that *luminous matter is fractal while radiation is isotropic*. A theory should put together these apparent conflicting facts. However if on one hand this problem may appear more difficult, on the other hand the broader framework of non-analytical structures may open new possibilities for its explanation. For example, we have stressed that isotropy and homogeneity are not so closed related for non-analytical distributions, We have shown in fact that the local isotropy implied by the Cosmological Principle is fully satisfied by a fractal structure even though this distribution is strongly inhomogeneous at all scales (Sylos Labini, 1994).

Another important point is the following: at a first sight one may think that the weakening of the experimental support for a homogenous metric may invalidate the Hubble law to estimate the distances form redshifts. However the Hubble law is an *experimental fact* and the homogenous metric corresponds to the simplest theoretical model that can describe it. Moreover, in the standard picture we find the following paradox. In the framework of the standard Freedman model the linearity of Hubble law is a consequence of homogeneity of matter distribution (Baryshev et al, 1994). If we have a strongly fluctuating behaviour for the density, we expect strong disturbances of the pure Freedman behavior. However the observation suggest the opposite however: a striking linearity of the redshift-distance relation is observed in the distance range $2 \div 25h^{-1}Mpc$, while at the same scales the luminous matter distribution is highly irregular and not homogenous. This suggest the possibility of explaining these facts needs a more complex model.

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